Orthogonal Functions

We shall now consider some of the important definitions regarding orthogonal functions.

(i) The *inner product* or *scalar product* of two functions $F(x)$ and $G(x)$ defined in the interval $a \leq x \leq b$, denoted as (F, G) or $(F|G)$, is

$$(F, G) = \int_a^b F^*(x)\, G(x)\, dx$$

The notation (F,G) for the scalar product of functions $F(x)$ and $G(x)$ is sometimes referred to as *bracket notation*.

(ii) Two functions $F(x)$ and $G(x)$ are *orthogonal* if their inner product is zero.

$$(F, G) = \int_a^b F^*(x)\, G(x)\, dx = 0$$

(iii) The *norm* of a function is defined by square root of inner product of the function with itself

$$N = (F, F)^{1/2} = \left[\int_a^b |F(x)|^2\, dx\right]^{1/2}$$

(iv) A function is *normalized* if its norm is unity.

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(iv) A function is *normalized* if its norm is unity.

$$(F, F)^{1/2} = \left[\int_a^b |F(x)|^2\, dx\right]^{1/2} = 1$$

or

$$(F, F) = \int_a^b F^*(x) F(x)\, dx = 1$$

where the integral on the right-hand side is called the *normalization integral*.

(v) Functions that are orthogonal and normalized are called *orthonormal functions*.

$$(F_i, F_j) = \delta_{ij}; \qquad i, j = 1, 2, \ldots$$

(vi) A set of functions $F_1(x)$, $F_2(x)$, $F_3(x)$, ... is *linearly dependent* if a relation of the type

$$\sum_i c_i F_i(x) = 0$$

exists, where the c_i's are not all zero. Otherwise they are *linearly independent*.

(vii) A set of linearly-independent functions $F_1(x)$, $F_2(x)$, ... is *complete*, if there is no other function which falls in the set of linearly-independent functions.

The *expansion theorem* states that any function $\phi(x)$ defined in the same interval can be expanded in terms of the set of linearly-independent functions as

$$\phi(x) = \sum_i c_i F_i(x)$$

The complete set need not be orthonormal.

The expansion of a function in terms of a complete orthonormal set of functions is of fundamental importance in quantum mechanics.

LINEAR OPERATOR

An *operator* can be defined as the rule by which a different function is obtained from any given function. Therefore, in

$$g(x) = \hat{A} f(x)$$

the operator $\hat{A}$ operating on $f(x)$ gives the function $g(x)$. So, in

$$g(x) = \hat{A} f(x) = [f(x)]^2$$

the operator squares the function $f(x)$. The operator $\hat{A}$ differentiates the function $f(x)$ with respect to x if

$$g(x) = \hat{A} f(x) = \frac{d}{dx} f(x)$$

An operator is said to be *linear* if it satisfies the relation

$$\hat{A}[c_1 f_1(x) + c_2 f_2(x)] = c_1 \hat{A} f_1(x) + c_2 \hat{A} f_2(x)$$

where c_1 and c_2 are constants. In

$$g(x) = \frac{d}{dx}\left[c_1 f_1(x) + c_2 f_2(x)\right] = c_1 \frac{d}{dx} f_1(x) + c_2 \frac{d}{dx} f_2(x)$$

the operator (d/dx) is linear. The operator which squares a function is not linear since

$$\begin{aligned} \hat{A}\,[c_1 f_1(x) + c_2 f_2(x)] &= [c_1 f_1(x) + c_2 f_2(x)]^2 \\ &= c_1^2 f_1^2 + c_2^2 f_2^2 + 2c_1 c_2 f_1 f_2 \\ &\neq c_1 f_1^2 + c_2 f_2^2 \end{aligned}$$

Linear operators are the most important ones in quantum mechanics and therefore we shall consider only such operators.

The sum and difference of operators $\hat{A}$ and $\hat{B}$ are defined by:

$$(\hat{A} \pm \hat{B}) f(x) = \hat{A} f(x) \pm \hat{B} f(x)$$

Addition is commutative:

$$\hat{A} + \hat{B} = \hat{B} + \hat{A}$$

Addition is associative:

$$(\hat{A} + \hat{B}) + \hat{C} = \hat{A} + (\hat{B} + \hat{C})$$

The product of two operators $\hat{A}$ and $\hat{B}$ is defined by:

$$\hat{A}\hat{B}\, f(x) = \hat{A}[\hat{B} f(x)]$$

Multiplication is associative:

$$\hat{A}(\hat{B} + \hat{C}) f(x) = (\hat{A}\hat{B} + \hat{A}\hat{C}) f(x)$$

Commutator of operators $\hat{A}$ and $\hat{B}$, denoted by $[\hat{A},\hat{B}]$, is defined as:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

It follows that

$$[\hat{A}, \hat{B}] = -\,[\hat{B}, \hat{A}]$$

If $\hat{A}\hat{B}\, f(x) = \hat{B}\hat{A}\, f(x)$, that is $[\hat{A},\hat{B}] = 0$, $\hat{A}$ and $\hat{B}$ are said to *commute*. If $\hat{A}\hat{B} + \hat{B}\hat{A} = 0$, $\hat{A}$ and $\hat{B}$ are said to *anticommute*.
$\hat{B}$ is usually denoted as $[\hat{A}, \hat{B}]_+$.

HERMITIAN OPERATOR

Let us consider two arbitrary functions $\psi_m(x)$ and $\psi_n(x)$. The operator A is said to be *Hermitian* if

$$\int_{-\infty}^{\infty} \psi_m^* A\psi_n \, dx = \int_{-\infty}^{\infty} (A\psi_m)^* \psi_n \, dx = \left(\int_{-\infty}^{\infty} \psi_n^* A\psi_m \, dx \right)^*$$

In bracket notation

$$(\psi_m, A\psi_n) = (A\psi_m, \psi_n) = (\psi_n, A\psi_m)^*$$

An operator is said to be **anti-Hermitian** if

$$(\psi_m, A\psi_n) = -(A\psi_m, \psi_n) = -(\psi_n, A\psi_m)^*$$

Two important theorems regarding Hermitian operators which we use throughout quantum mechanics are the following.

Theorem 1 The eigenvalues of Hermitian operators are real.

Proof Consider a Hermitian operator A. Its eigenvalue equation be

$$A\psi_n = a_n\psi_n$$

Taking inner product with ψ_n, we get

$$(\psi_n, A\psi_n) = a_n(\psi_n, \psi_n) = a_n$$

Since A is Hermitian, we have

$$(\psi_n, A\psi_n) = (A\psi_n, \psi_n) = a_n^* (\psi_n, \psi_n) = a_n^*$$

It follows that $a_n = a_n^*$ which is possible only when a_n is real. Real eigenvalues of Hermitian operators play a very important role in quantum mechanics.

Theorem 2 Any two eigenfunctions of a Hermitian operator that belong to different eigenvalues are orthogonal.

Proof Let ψ_m and ψ_n be the eigenfunctions of the operator A corresponding to the eigenvalues a_m and a_n respectively. Then

$$A\psi_m = a_m\psi_m, \qquad A\psi_n = a_n\psi_n$$

we obtain

$$(\psi_n, A\psi_m) = a_m(\psi_n, \psi_m)$$

Since operator A is Hermitian,

$$(A\psi_n, \psi_m) = a_m(\psi_n, \psi_m) \quad \text{or} \quad a_n(\psi_n, \psi_m) = a_m(\psi_n, \psi_m)$$

or

$$(a_n - a_m)(\psi_n, \psi_m) = 0$$

As $a_n \neq a_m$, we have

$$(\psi_n, \psi_m) = 0$$

Hence, the eigenfunctions ψ_n and ψ_m are orthogonal.

Schmidt Orthogonalization Procedure

Let ψ_i and ψ_j be two normalized eigenfunctions of the Hermitian operator A having the same eigenvalue a. Then

$$A\psi_i = a\psi_i, \qquad A\psi_j = a\psi_j$$

The linear combination of ψ_i and ψ_j is given by

$$\psi_k = c_1\psi_i + c_2\psi_j$$

where c_1 and c_2 are constants. Also,

$$A\psi_k = A(c_1\psi_i + c_2\psi_j) = a(c_1\psi_i + c_2\psi_j)$$

that is, ψ_k is also an eigenfunction of the operator A with the same eigenvalue a.

We assume that

$$(\psi_i, \psi_k) = 0 \quad \text{and} \quad (\psi_k, \psi_k) = 1$$

Then

$$(\psi_i, \psi_k) = 0 \quad \text{gives} \quad c_1 = -c_2(\psi_i, \psi_j)$$

and

$$(\psi_k, \psi_k) = 1 \quad \text{gives} \quad c_1^2 + c_2^2 + c_1c_2[(\psi_i, \psi_j) + (\psi_j, \psi_i)] = 1$$

The constants are assumed to be real. From Eqs. we have

$$c_2 = \frac{1}{\sqrt{1 - |(\psi_i, \psi_j)|^2}} \quad \text{and} \quad c_1 = -\frac{(\psi_i, \psi_j)}{\sqrt{1 - |(\psi_i, \psi_j)|^2}}$$

Hence

$$\psi_k = \frac{\psi_j - (\psi_i, \psi_j)\psi_i}{\sqrt{1 - \left|(\psi_i, \psi_j)\right|^2}}$$

This ψ_k is a normalized eigenfunction of the operator A corresponding to the eigenvalue a. It is orthogonal to ψ_i. This procedure is a case of *Schmidt orthogonalization* procedure for systems having two-fold degeneracy. Similar procedure can be followed for higher order degenerate cases.

POSTULATES OF QUANTUM MECHANICS

Postulate 1: Wave function

The state of a system having n degrees of freedom can be completely specified by a function Ψ of coordinates $q_1, q_2, \ldots, q_n$ and time t which is called the *wave function* or *state function* or *state vector* of the system. Ψ and its derivatives must be continuous, finite and single valued over the domain of the variables of Ψ. All possible information about the system can be derived from this wave function.

The wave function Ψ as such is not an observable, but in some way it is related to the presence of the particle.

The representation in which the wave function is a function of coordinates and time is called the *coordinate representation*. In the *momentum representation*, the wave functions are functions of the momentum components and time.
the coordinate representation.

Postulate 2: Operators

Classical observables and their quantum mechanical operators

Observable	*Classical form*	*Operator in coordinate representation*
Coordinates	x, y, z	x, y, z
Function of coordinate	$f(x, y, z)$	$f(x, y, z)$
Momentum components	p_x, p_y, p_z	$-i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}$
Momentum	$\mathbf{p}$	$-i\hbar\nabla$
Energy	E	$i\hbar\frac{\partial}{\partial t}$

Operators representing some of the other dynamical variables take the following form:

Kinetic energy operator. For a particle of mass m and momentum $\mathbf{p}$, the kinetic energy

$$T = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

$$T = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = -\frac{\hbar^2}{2m}\nabla^2$$

Hamiltonian operator. For a particle of mass m moving in a potential $V(x, y, z)$, the Hamiltonian

$$H = \frac{p^2}{2m} + V(x, y, z)$$

or

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(x, y, z)$$

Postulate 3: Expectation Value

When a system is in a state described by a wave function Ψ, the expectation value of any observable A is given by

$$\langle A \rangle = \int_{-\infty}^{\infty} \Psi^* A \Psi \, d\tau$$

where A in the integral is the operator associated with the observable A. the wave function Ψ is assumed to be normalized. If the wave function is not normalized

$$\langle A \rangle = \frac{\int_{-\infty}^{\infty} \Psi^* A \Psi \, d\tau}{\int_{-\infty}^{\infty} \Psi^* \Psi \, d\tau}$$

the sandwiching of the operator between Ψ^* and Ψ is a necessity.

Postulate 4: Eigenvalues

Often an operator A operating on a function multiplies the function by a constant.

$$A\psi(x) = a\psi(x)$$

where a is a constant with respect to x. The function $\psi(x)$ is called the *eigenfunction* of the operator A corresponding to the *eigenvalue* a. In

$$\frac{de^{kx}}{dx} = k\,e^{kx}$$

e^{kx} is an eigenfunction of the operator d/dx corresponding to the eigenvalue k.

Postulate 5: Time Development of a Quantum System

The time development of a quantum system can be described by the evolution of state function in time by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi(\mathbf{r},t)}{\partial t} = H\Psi(\mathbf{r}, t)$$

where H is the Hamiltonian operator of the system which is independent of time.

This procedure of considering the state function depends on coordinates and time and the operator to be independent of time is called the *Schrödinger picture* or *Schrödinger representation.*

SIMULTANEOUS MEASURABILITY OF OBSERVABLES

We have been discussing the measurement of one observable at a time. If two observables are simultaneously measurable in a particular state of a given system, then the state function is an eigenfunction of both the operators. Two observables are said to be *compatible*, if their operators have a common set of eigenfunctions. The following two theorems indicate the connection between compatible observables and commuting operators.

Theorem Operators having common set of eigenfunctions commute.

Proof Consider operators A and B with the common set of eigenfunctions ψ_i, $i = 1, 2, \ldots$ as

$$A\psi_i = a_i \psi_i, \quad \text{and} \quad B\psi_i = b_i\psi_i$$

Then

$$AB\psi_i = A(b_i\psi_i) = b_iA\psi_i = a_ib_i\psi_i$$

and

$$BA\psi_i = B(a_i\psi_i) = a_iB\psi_i = a_ib_i\psi_i$$

Since $AB\psi_i = BA\psi_i$, A commutes with B. Hence the result.

Theorem Commuting operators have common set of eigenfunctions.

Proof Consider two commuting operators A and B. The eigenvalue equation for A be

$$A\psi_i = a_i\psi_i, \qquad i = 1, 2, \ldots$$

Operating both sides from left by B

$$BA\psi_i = a_iB\psi_i$$

Since B commutes with A

$$A(B\psi_i) = a_i(B\psi_i)$$

That is, $B\psi_i$ is an eigenfunction of A with the same eigenvalue a_i. If A has only nondegenerate eigenvalues, $B\psi_i$ can differ from ψ_i only by a multiplicative constant, say b_i

$$B\psi_i = b_i\psi_i$$

In other words ψ_i is a simultaneous eigenfunction of both A and B.

DIRAC'S NOTATION

The state of a system can be represented by a vector called *state vector* in the vector space. Dirac introduced the symbol $|\ \rangle$, called the *ket vector* or simply *ket* to denote a state vector which will take different forms in different representations. To distinguish the ket vectors corresponding to different states, a label is introduced in the ket. Thus, the state vector corresponding to $\psi_a(\mathbf{r})$ is denoted by the ket $|a\rangle$. Corresponding to every vector, $|a\rangle$ is defined a conjugate vector $|a\rangle^*$ for which Dirac used the notation $\langle a|$ which is called a *bra vector* or simply *bra*. The conjugate of a ket vector is a bra vector and vice versa. A scalar in the ket space becomes its complex conjugate in the bra space. The bra–ket notation is a distorted form of the *bracket* notation. Thus, the bracket symbol $(\ |\ )$ is distorted to $\langle\ |$ and $|\ \rangle$ in the Dirac notation. The words 'bra' and 'ket' were derived from the word bracket by dropping the letter *'c'*.

Operation by an operator A on a ket vector produces another ket vector.

$$A|a\rangle = |a'\rangle$$

Operation on a bra vector from the right by A gives another bra vector

$$\langle b|A = \langle b'|$$

In terms of bra and ket vectors, the definition of the inner product of the state vectors ψ_a and ψ_b takes the form

$$(\psi_a, \psi_b) = \int \psi_a^* \psi_b \, d\tau = \langle a|b\rangle$$

The *norm* of a ket $|a\rangle$, denoted by $\langle a|a\rangle$ is a real nonnegative number. That is

$$\langle a|a\rangle \geq 0$$

The equality sign holds only if $|a\rangle = 0$. The ket $|a\rangle$ is said to be *normalized* if

$$\langle a|a\rangle = 1$$

Kets $|a\rangle$ and $|b\rangle$ are *orthogonal* if

$$\langle a|b\rangle = 0$$

The *orthonormality relation* is expressed as

$$\langle a_i|a_j\rangle = \delta_{ij}$$

In this notation, the condition for an operator to be Hermitian is

$$\langle a|A|b\rangle = \langle b|A|a\rangle^*$$

Compared to conventional notation, Dirac's notation is compact.

EQUATION OF MOTION

The state vector changes with time but the operator remains constant (*Schrödinger representation* or *Schrödinger picture*),

Schrödinger Representation

We are very familiar with the wave mechanical approach to quantum mechanics and therefore it is appropriate to start with the Schrödinger representation.

in this picture the state vectors are time-dependent kets $|\psi_s(t)\rangle$ and the operators are constants in time. The equation of motion is then an equation for $|\psi_s(t)\rangle$, the subscript 's' is to indicate Schrödinger picture. The ket $|\psi_s(t)\rangle$ varies in accordance with the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi_s(t)\rangle = H|\psi_s(t)\rangle$$

As the Hamiltonian H is independent of time, Eq. can be integrated to give

$$|\psi_s(t)\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|\psi_s(0)\rangle$$

Here, the operator exp $(-iHt/\hbar)$ is defined as

$$\exp\left(-\frac{iHt}{\hbar}\right) = \sum_{n=0}^{\infty} \frac{(-iHt/\hbar)^n}{n!}$$

Equation reveals that the operator $\exp(-iHt/\hbar)$ changes the ket $|\psi_s(0)\rangle$ into ket $|\psi_s(t)\rangle$. Since H is Hermitian and t is real, this operator is unitary and the norm of the ket remains unchanged. The Hermitian adjoint of Eq. is

$$-i\hbar\frac{d}{dt}\langle\psi_s(t)| = \langle\psi_s(t)|H^\dagger = \langle\psi_s(t)|H$$

whose solution is

$$\langle\psi_s(t)| = \langle\psi_s(0)|\exp\left(\frac{iHt}{\hbar}\right)$$

Next we consider the time derivative of expectation value of the operator A_s. The time derivative of $\langle A_s\rangle$ is given by

$$\frac{d}{dt}\langle A_s\rangle = \frac{d}{dt}\langle\psi_s(t)|A_s|\psi_s(t)\rangle$$

where A_s is the operator representing the observable A. Replacement of the factors

$$\frac{d}{dt}|\psi_s(t)\rangle \quad \text{and} \quad \frac{d}{dt}\langle\psi_s(t)|$$

using Eqs.

$$\frac{d}{dt}\langle A_s\rangle = \frac{1}{i\hbar}\langle\psi_s(t)|A_sH - HA_s|\psi_s(t)\rangle + \langle\psi_s(t)|\frac{\partial A_s}{\partial t}|\psi_s(t)\rangle$$

or

$$\frac{d}{dt}\langle A_s\rangle = \frac{1}{i\hbar}[A_s, H] + \left\langle\frac{\partial A_s}{\partial t}\right\rangle$$

If A_s has no explicit dependence on time, we get

$$i\hbar\frac{d}{dt}\langle A_s\rangle = [A_s, H]$$

If the operator A_s commutes with the Hamiltonian, it is a constant in time.

Dirac Delta Function

The Dirac delta function $\delta(x)$ is not a function in the usual mathematical sense. In the normal sense a function will have a definite value for each point in its domain. The delta function $\delta(x)$ acquires a meaning only when it appears in an integral. *The Dirac delta function* $\delta(x)$ is defined by the conditions:

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \tag{C.1}$$

such that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \tag{C.2}$$

By making a change of origin, we can write Eq. (C.1) as

$$\delta(x - x_0) = \begin{cases} 0, & x \neq x_0 \\ \infty, & x = 0 \end{cases} \tag{C.3}$$

such that

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1 \tag{C.4}$$

If $f(x)$ is an arbitrary function, well defined at $x = 0$, then the integration of $f(x)$ with the delta function selects the value of $f(x)$ at the origin

$$\int f(x)\delta(x)\ dx \ = \ f(0) \tag{C.5}$$

Here, the integration is over the domain in which $f(x)$ is defined, provided the range includes the origin. It follows from Eq. (C.2) that

$$\int f(x)\delta(x - x_0)\ dx = f(x_0) \tag{C.6}$$

where the range of integration must include the point $x = x_0$.

Generalization of Dirac delta function to three-dimensional space is straightforward. If $\mathbf{r}$ is the position vector with components x, y, z, then the three-dimensional delta function:

$$\delta(\mathbf{r} - \mathbf{r}_0) \ = \ \delta(x - x_0)\ \delta(y - y_0)\ \ \delta(z - z_0) \tag{C.7}$$

such that

$$\int f(\mathbf{r})\delta\ (\mathbf{r} - \mathbf{r}_0)\ d\mathbf{r} = f(\mathbf{r}_0) \tag{C.8}$$

where the range of integration includes the point (x_0, y_0, z_0).

Properties of the Delta Function

(i) The delta function is an even function: $\delta(-x) = \delta(x)$

(ii) $x\delta(x) = 0$

(iii) $x\delta(x - x_0) = x_0\ \delta(x - x_0)$

(iv) $f(x)\ \delta(x - x_0) = f(x_0)\ \delta(x - x_0)$

(v) $$\delta(ax) = \frac{1}{a}\delta(x), \qquad a > 0 \tag{C.9}$$

(vi) $$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x - a) + \delta(x + a)]$$

(vii) $$\int \delta(x - b)\ \delta(a - x)\, dx = \delta(a - b)$$

Representation of Delta Function

Mathematically, delta function can be considered as the limit of a function which becomes more peaked at the origin when a parameter approaches zero.

Commutator Algebra

The *commutator* of two operators $\hat{A}$ and $\hat{B}$, denoted by $[\hat{A},\ \hat{B}]$, is defined by

$$\boxed{[\hat{A},\ \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A},}$$

and the *anticommutator* $\{\hat{A},\ \hat{B}\}$ is defined by

$$\boxed{\{\hat{A},\ \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}.}$$

Two operators are said to commute if their commutator is equal to zero and hence $\hat{A}\hat{B} = \hat{B}\hat{A}$. Any operator commutes with itself:

$$[\hat{A},\ \hat{A}] = 0.$$

Note that if two operators are Hermitian and their product is also Hermitian, these operators commute:

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger = \hat{B}\hat{A},$$

and since $(\hat{A}\hat{B})^\dagger = \hat{A}\hat{B}$ we have $\hat{A}\hat{B} = \hat{B}\hat{A}$.

Properties of commutators

Using the commutator relation (2.79), we can establish the following properties:

- Antisymmetry:

$$[\hat{A},\ \hat{B}] = -[\hat{B},\ \hat{A}]$$

- Linearity:

$$[\hat{A},\ \hat{B} + \hat{C} + \hat{D} + \cdots] = [\hat{A},\ \hat{B}] + [\hat{A},\ \hat{C}] + [\hat{A},\ \hat{D}] + \cdots$$

- Hermitian conjugate of a commutator:

$$[\hat{A},\ \hat{B}]^\dagger = [\hat{B}^\dagger,\ \hat{A}^\dagger]$$

- Distributivity:

$$[\hat{A},\ \hat{B}\hat{C}] = [\hat{A},\ \hat{B}]\hat{C} + \hat{B}[\hat{A},\ \hat{C}]$$
$$[\hat{A}\hat{B},\ \hat{C}] = \hat{A}[\hat{B},\ \hat{C}] + [\hat{A},\ \hat{C}]\hat{B}$$

- Jacobi identity:

$$[\hat{A}, [\hat{B},\ \hat{C}]] + [\hat{B},\ [\hat{C},\ \hat{A}]] + [\hat{C},\ [\hat{A},\ \hat{B}]] = 0$$

- Operators commute with scalars: an operator $\hat{A}$ commutes with any scalar b:

$$[\hat{A},\ b] = 0$$

Momentum Representation

The basis $\{|\ \vec{p}\rangle\}$ of the momentum representation is obtained from the eigenkets of the momentum operator $\hat{\vec{P}}$:

$$\hat{\vec{P}}\ |\ \vec{p}\rangle\ =\ \vec{p}\ |\ \vec{p}\rangle,$$

where $\vec{p}$ is the momentum vector. The algebra relevant to this representation can be easily inferred from the position representation. The orthonormality and completeness conditions of the momentum space basis $|\ \vec{p}\rangle$ are given by

$$\langle\vec{p}\ |\ \vec{p}\,'\rangle\ =\ \delta(\vec{p}-\vec{p}\,')\qquad\text{and}\qquad\int \mathrm{d}^3\mathrm{p}\ |\ \vec{\mathrm{p}}\rangle\langle\vec{\mathrm{p}}\ |=\hat{\mathrm{I}}.$$

Expanding $|\ \psi\rangle$ in this basis, we obtain

$$|\ \psi\rangle=\int d^3p\ \ |\ \vec{p}\rangle\langle\vec{p}\ |\ \psi\rangle=\int d^3p\ \Psi(\vec{p})\ |\ \vec{p}\rangle,$$

where the expansion coefficient $\Psi(\vec{p})$ represents the *momentum space wave function*. The quantity $|\ \Psi(\vec{p})\ |^2\ d^3p$ is the probability of finding the system's momentum in the volume element d^3p located between $\vec{p}$ and $\vec{p}+d\vec{p}$.

the scalar product between two states is given in the momentum space by

$$\langle\phi\ |\ \psi\rangle\ =\ \langle\phi\ |\left(\int d^3p\ \ |\ \vec{p}\rangle\langle\vec{p}\ |\right)|\ \psi\rangle=\int d^3p\ \Phi^*(\vec{p})\Psi(\vec{p}).$$

Since $\hat{\vec{P}}\ |\ \vec{p}\rangle\ =\ \vec{p}\ |\ \vec{p}\rangle$ we have

$$\langle\vec{p}\,'\ |\ \hat{\vec{P}}^{\,n}\ |\ \vec{p}\rangle=\vec{p}^{\,n}\delta(\vec{p}\,'-\vec{p}).$$